# ON THE HOMOTOPY TYPE OF PRINCIPAL CLASSICAL GROUP BUNDLES OVER SPHERES

BY

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ABSTRACT

The total spaces of principal $SU(n-1)$ bundles over $S^{2n-1}$ are classified. The classification of $Sp(n-1)$ bundles over $S^{4n-1}$ is studied as well. As an intermediate step the homotopy equivalences of $SU$ and $Sp$ are classified.

## 0. Introduction

In their study of the non-cancellation phenomenon of cartesian products Hilton and Roitberg [3] investigated the homotopy type of principal $S^3$ bundles over spheres. From the point of view of the theory of finite $CW$ $H$-spaces the most interesting case was that of principal $S^3$ bundles over $S^7$ as the bundle classified by $7w \in \pi_6(S^3)$ turned out to be a newly discovered $H$-space whose homotopy type is different from that of $Sp(2)$ and $S^3 \times S^7$ (the only previously known $H$-spaces of type 3, 7).

Since then, other finite $CW$-$H$-spaces which are principal classical group bundles over spheres were discovered: [2], [4], and [7]. Combining the results of [6] and [7] one has:

THEOREM 0.1. *Let $G(n,d), d$ be one of the following: $SO(n), 1$, $SU(n), 2$, or $Sp(n), 4$. Let $\alpha = (G(n-1,d) \to G(n,d) \xrightarrow{f} S^{dn-1})$ be the classical fiber bundle* ($dn-1$ –odd). *Denote by $M(n,d,\lambda)$ the total space of the fiber bundle $h_\lambda^*(\alpha)$ induced by a map $h_\lambda\colon S^{dn-1} \to S^{dn-1}$ of degree $\lambda$. Then:*

(a) *If $\lambda$ is odd then $M(n,d,\lambda)$ is an $H$-space.*

(b) *If $nd-1 \neq 3,7$ and $M(n,d,\lambda)$ admits an $H$-structure then $\lambda$ is odd.*

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From the point of view of the classification problem of finite $CW$-$H$-spaces one would like to know how many homotopy types do the $\{M(n,d,\lambda)\}$ represent. This question of course amounts to the very fundamental problem of classifying the total spaces of principal $G(n-1,d)$ bundles over $S^{dn-1}$.

In this note we give a complete answer for that question in case $d = 2$ and a partial one for $d = 4$.

THEOREM A. (a) *If* $d = 2$ *or* 4 *and* $M(n,d,\lambda) \approx M(n,d,\lambda')$ then $\lambda \equiv \pm\lambda' \bmod (dn/2-1)!$

(b) *If* $\lambda \equiv \pm\lambda' \bmod k(n,d)$ *then* $M(n,d,\lambda) \approx M(n,d,\lambda')$ *where* $k(n,d)$ *is the order of the cyclic group* $\pi_{dn-2}(G(n-1,d))$:

$$k(n,2) = (n-1)!$$

$$k(n,4) = \begin{cases} (2n-1)! & \text{if } n \text{ is odd} \\ 2[(2n-1)!] & \text{if } n \text{ is even.} \end{cases}$$

Note that if $d = 2$ or $n$ is odd (a) and (b) give a necessary and sufficient condition for $M(n,d,\lambda) \approx M(n,d,\lambda')$.

COROLLARY. *Suppose* $dn > 8$. *If* $d = 2$ *or* $n$ *is odd* $d = 4$ *then there exist exactly* $\frac{1}{4}(dn/2-1)!$ *different homotopy types of finite CW-H-spaces among the total spaces of principal* $G(n-1,d)$ *bundles over* $S^{dn-1}$. *There are at least that number if* $d = 4$, *n-even.*

One should note that Theorem A, part (b) is a direct consequence of the classical theorems of Feldbau and Steenrod ([5, theorems 18.5, p. 99 and 19.3, p. 101]).

The relation of Theorem A to the non-cancellation phenomena can be expressed by the following:

THEOREM B. *If* $\lambda \equiv \pm\lambda' \bmod k(n,d)$, *λ-odd, then*

$$M(n,d,\lambda) \times S^{dn-1} \approx M(n,d,\lambda') \times S^{dn-1}.$$

## 1. The homotopy equivalences of $SU$

The following facts concerning the homotopy groups of $SU$, $SU(n)$, and $Sp(n)$ are classical theorems of Bott [1].

THEOREM 1. 1. (Bott [1, p. 314–315])

(a) $\pi_{2k+1}(SU) = Z$, $k \geqq 1$

$\pi_{2k}(SU) = 0$

(b) $\pi_{4k+3}(Sp) = Z, \ \pi_{8k+4}(Sp) = \pi_{8k+5}(Sp) = Z_2$

$\pi_{4k+2}(Sp) = \pi_{8k+1}(Sp) = \pi_{8k+2}(Sp) = 0$

(c) $\pi_k(SU(n)) = \pi_k(SU)$ if $k < 2n$

$\pi_{2n}(SU(n)) = Z_{n!}$

(d) There exists a fibration of infinite loop spaces and maps

$$Sp \xrightarrow{\phi} SU \xrightarrow{\psi} BBSp$$

where $BBSp$ is the classifying space of the classifying space of $Sp$. As a consequence one can obtain:

COROLLARY 1.2. (a) *The degree of the Hurewicz-Serre homomorphism* $Z = \pi_{2k-1}(SU) = \pi_{2k-1}/torsion \to PH_{2k-1}(SU,Z) = PH_{2k-1}(SU,Z)/torsion$ *is* $(k-1)!$

(b) *The degree of* $\pi_{4n-1}(\phi)$: $\pi_{4n-1}(Sp) \to \pi_{4n-1}(SU)$ is 1 for *n odd* and 2 *for n even. The degree of* $\pi_{4n+1}(\psi)$: $\pi_{4n+1}(SU) \to \pi_{4n+1}(BBSp)$ *is* 1 *for n even and* 2 *for n odd,* $n > 1$.

(c) $\pi_{4n+2}(Sp(n))$ *is cyclic of order* $(2n+1)!$ *if n is even and of order* $2[(2n+1)!]$ *if n is odd.*

(d) *The degree of the Hurewicz-Serre homomorphism* $Z = \pi_{4n-1}(Sp)/torsion \to PH_{4n-1}(Sp)/torsion = Z$ *is* $(2n-1)!$ *if n is odd and* $2[(2n-1)!]$ *if n is even.*

For a $CW$ complex $Y$ let $r_n\colon Y \to Y_n$ be its Postnikov approximation in dimension $\leqq n$, i.e.: $\pi_k(r_n)$ is an isomorphism if $k \leqq n$ and $\pi_k(Y_n) = 0$ if $k > n$.

LEMMA 1.3. *Let Y be a CW complex. If* $H^*(Y,Q)$ *is a free (associative commutative graded)) algebra then*

(a) $r_n^*\colon H^*(Y_n,Q) \to H^*(Y,Q)$ *is a monomorphism,* $\operatorname{im} r_n^*$ *being the subalgebra of* $H^*(Y,Q)$ *generated by* $\sum_{k\leqq n} H^k(Y,Q)$.

(b) *The k-invariants of Y are of finite order.*

PROOF. (a) Let $A \subset H^*(Y,Z)$ be a free graded subgroup so that the composition $A \otimes Q \to H^*(Y,Z) \otimes Q \to H^*(Y,Q) \to QH^*(Y,Q)$ is an isomorphism (where $QH^*(Y,Q)$ is the module of indecomposables of $H^*(Y,Q)$). Then there exists a map $\psi\colon Y \to K(A) = \prod_n K(A \cap H^n(Y,Z), n)$ realizing $A$ and therefore yielding an isomorphism of rational cohomology. Hence, $\pi(\psi) \otimes Q\colon \pi(Y) \otimes Q \approx \pi(K(A)) \otimes Q$ and consequently $\pi(\psi_n) \otimes Q\colon \pi(Y_n) \otimes Q \approx \pi(K(A^{(n)})) \otimes Q$ where $K(A^{(n)}) = \prod_{m \leqq n} K(A \cap H^m(Y,Z), m)$ and $\psi_n\colon Y_n \to K(A^{(n)}) = [K(A)]_n$ is the Postnikov approximation of $\psi$. Thus, $\psi_n$ induces an isomorphism of rational cohomology and as im $[H^*(K(A^{(n)}), Q) \to H^*(K(A), Q)]$ represents the subalgebra generated by $\sum_{m \leqq n} H^n(K(A),Q)^m$ (a) follows.

(b) By (a) $H^*(Y_n,Q) \to H^*(Y,Q)$ is a monomorphism and, consequently, so is $H^*(r_{n,n-1},Q)\colon H^*(Y_{n-1},Q) \to H^*(Y_n,Q)$ where $Y_n \xrightarrow{r_{n,n-1}} Y_{n-1} \xrightarrow{k_{n-1}} K(\pi_n(Y),n+1)$ is the Postnikov fibration. As $\operatorname{im} H^*(k_{n-1},Q) \subset \ker H^*(r_{n,n-1}Q) = 0$, $H^*(k_{n-1},\mathrm{Q}) = 0$, and $\operatorname{im}(H^*(k_{n-1},Z))$, (i.e., the integral $k$-invariants) are of finite order and (b) follows.

Throughout this section we shall consider the following properties of a pair of $CW$ complexes $X$ and $Y$:

(a) $H^*(X,Z)$ and $\pi_*(Y)$ are torsion free and $H^*(Y,Q)$ is a free algebra.

(b) $Y$ is a homotopy associative $H$-space.

(c) $H^*(X,Z)$ is a free algebra.

(d) $\operatorname{rank} QH^m(X,Z) \leqq 1$ for all $m \geqq 0$.

Lemma 1.4. *If* (a) *is satisfied then*

$$r_{n*}\colon [X,Y] \to [X,Y_n]$$

*is onto.*

Proof. By (a) and 1.3 all $k$-invariants of $Y$ are integral and of finite order. As $H^*(X,Z)$ has no (non-zero) elements of finite order every map $X \to Y_n$ can be lifted (up to homotopy) to a map $X \to Y$.

Lemma 1.5. *If* (a) *is satisfied and* $\phi\colon X' \to X$ *yields an epimorphism of integral cohomology then* $\phi^*\colon [X,Y] \to [X',Y]$ *is onto.*

Proof. Given $f\colon X' \to Y$ and suppose, inductively, the following (homotopy) commutative diagram:

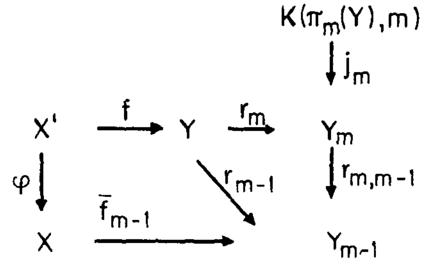



Fig. 1

(As $Y_0$ is a point such a diagram exists for $m = 1$.)

By 1.4 $\tilde{f}_{m-1}$ can be lifted to $\tilde{f}_m : X \to Y_m$, $r_{m,m-1} \circ \tilde{f}_m = \tilde{f}_{m-1}$. As $r_{m,m-1} \circ \tilde{f}_m \circ \phi \approx r_{m,m-1} \circ r_m \circ f$ and as $r_{m,m-1}$ can be considered as a principal fibration $[X', K(\pi_m(Y), m)]$ acts on $[X', Y_m]$ and one gets: $[r_m \circ f] = [w] \cdot [\tilde{f}_m \circ \phi]$ for some $w : X' \to K(\pi_m(Y), m)$. The fact that $H^*(\phi, Z)$ is onto is equivalent to the extendability of $w$ to $X$: $w \approx \bar{w} \circ \phi$ for some $\bar{w}: X \to K(\pi_m(Y), m)$. Let $[\bar{f}_m] = [\bar{w}] \cdot [\tilde{f}_m]$, $\bar{f}_m : X \to Y_m$. Then $[r_{m,m-1} \circ \bar{f}_m] = r_{m,m-1*}[\bar{f}_m] = r_{m,m-1*}[\bar{w}] \cdot [\tilde{f}_m] = [r_{m,m-1} \circ \tilde{f}_m]$. Hence, $r_{m,m-1} \circ \bar{f}_m \approx \bar{f}_{m-1}$ and also

$$[\bar{f}_m \circ \phi] = \phi^*[\bar{f}_m] = \phi^*[\bar{w} \cdot \tilde{f}_m] = [\bar{w} \circ \phi] \cdot [\tilde{f}_m \circ \phi] = [w][\tilde{f}_m \circ \phi] = [r_m \circ f]$$

and one obtains the following homotopy commutative diagram:

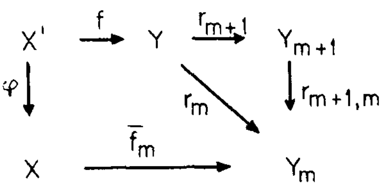



Fig. 2

Thus, $\bar{f} = \lim \bar{f}_m$ satisfies $\bar{f} \circ \phi \approx f$ and $\phi^*$ is onto.

PROPOSITION 1.6. *If* (a), (*b*), *and* (*c*) *are satisfied, then*

$$[X \wedge X, Y] \xrightarrow{\bar{\Delta}^*} [X, Y] \xrightarrow{\pi} \mathrm{Hom}(\pi(X), \pi(Y))$$

*is exact, where* $\bar{\Delta}: X \to X \wedge X$ *is the composition* $X \xrightarrow{\Delta} X \times X \xrightarrow{\Lambda} X \wedge X$ ($\Delta$ — *the diagonal,* $\Lambda$ — *the identification map*).

In order to prove 1.6 we first prove the following:

LEMMA 1.6.1. *Let* $f: T_1 \to T_2$, $T_2$ $m-1$ *connected,* $\pi_m(T_2)$ *free and* $H^*(T_1, Z)$ *is a free algebra in* dim $\leqq m$. *If* $\pi_m(f) = 0$ *then* $\mathrm{im}\, H^m(f, Z)$ *lies in the submodule of decomposable elements of the algebra* $H^*(T_1, Z)$.

PROOF. One may assume that $\pi_k(T_i) = 0$ for $k > m$, $i = 1,2$; otherwise $T_i$ and $f$ can be replaced by their Postnikov approximations without affecting the homotopy and cohomology in dim $\leqq m$. Hence, $T_2 = K(\pi_m(T_2), m)$ and one has $j: K(\pi_m(T_1), m) \to T_1$ yielding an isomorphism $\pi_m(j)$. $\pi_m(f) = 0$ is equivalent to the fact that the following composition is null homotopic:

$$K(\pi_m(T_1), m) \xrightarrow{j} T_1 \xrightarrow{f} T_2 = K(\pi_m(T_2), m).$$

By 1.3 $\psi_0: T_1 \to \Pi_{k=1}^{m} K((\pi_k(T_1)/\text{torsion}),k) = K_0$ yields an isomorphism of rational cohomology. In particular,

$$QH^m(j,Q): QH^m(T_1,Q) \overset{\approx}{\to} QH^m(K(\pi_m(T_1), m),Q)$$

and $\ker H^m(j,Q)$ lies in the submodule of decomposables of $H^*(T_1,Q)$. Hence, $\operatorname{im} H^m(f,Q) \subset \ker H^m(j,Q)$ lies in the module of decomposable and the freeness of the algbera $H^*(T_1, Z)$ in dim $\leqq m$ implies that $QH^m(T_1, Z) \to QH^m(T_{,1}Q)$ is a monomorphism and $\operatorname{im} H^m(f, Z)$ lies in the module of decomposables of $H^m(T_1, Z)$.

Proof of 1.6. As $\Omega\Lambda \approx *$, $\pi(\Lambda) = \pi(\bar{\Delta}) = 0$ and $\pi \circ \bar{\Delta}^* = 0$. Suppose $f \in \ker \pi$. Let $i_m: Y^m \to Y$ be the $m-1$ connective fibering of $Y$. Suppose inductively that there exists $f_m: X \to Y^m$ so that $[f] - [i_m \circ f_m] \in \operatorname{im} \bar{\Delta}^*$. As $\pi(\operatorname{im} \bar{\Delta}^*) = 0$ and $\pi(i_m)$ is a monomorphism, $\pi(f) = 0$ implies $\pi(f_m) = 0$. Let $r: Y^m \to K(\pi_m(Y),m)$ be the Postnikov approximation of $Y^m$. By 1.6.1 $\operatorname{im} H^*(f_m, Z)$ lies in the module of decomposables and hence $r \circ f_m$ can be factored as demonstrated in the following (homotopy) commutative diagram:

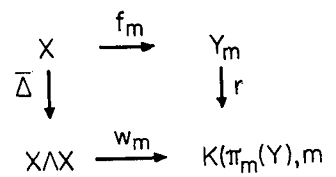



Fig. 3

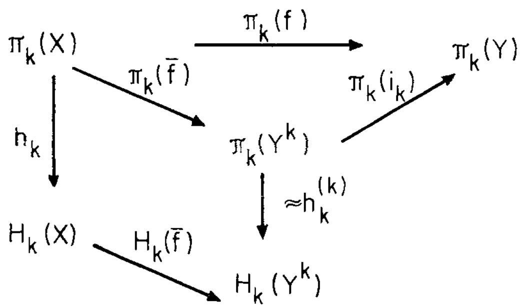
As $H^*(X \wedge X, Z)$, $\pi_*(Y^m)$ are torsion free and $H^*(Y^m,Q) \approx H^*(Y,Q)/\operatorname{im} H^*(r_m,Q)$ is by 1.3 a free algebra, by 1.4 $w_m$ can be lifted to $\bar{w}_m: X \wedge X \to Y^m$, $r \circ \bar{w}_m \approx w_m$.

Put $[\tilde{f}_{m+1}] = [f_m] - [\bar{w}_m \circ \bar{\Delta}]$, then $[r \circ \tilde{f}_{m+1}] = [r \circ f_m] - [r \circ \bar{w}_m \circ \bar{\Delta}]$ $= [r \circ f_m] - [w_m \circ \bar{\Delta}] = 0$. Hence, $\tilde{f}_{m+1}: X \to Y^m$ can be lifted to $f_{m+1}: X \to Y^{m+1}$, $[i_m \circ f_m] - [i_{m+1} \circ f_{m+1}] = [i_m \circ f_m] - [i_m \circ \tilde{f}_{m+1}] = [i_m \circ \bar{w}_m \circ \bar{\Delta}] \in \operatorname{im} \bar{\Delta}^*$ and consequently $[f] - [i_{m+1} \circ f_{m+1}] \in \operatorname{im} \bar{\Delta}^*$. Passing to a limit, one gets $[f] \in \operatorname{im} \bar{\Delta}^*$.

Proposition 1.7. *Suppose* (a), (b), *and* (c) *hold. Let* $f: X \to Y$, $\pi_m(f) = 0$ *for* $m < k$. *Then*
$\pi_k(f) \in \operatorname{im}[\operatorname{Hom}(H_k(X), \pi_k(Y)) \xrightarrow{h_k^*} \operatorname{Hom}(\pi_k(X), \pi_k(Y))]$ *where* $h_k^*$ *is induced by the Hurewicz homomorphism* $h_k: \pi_k(X) \to H_k(X) = H_k(X, Z)$.

Proof. Let $r_{k-1}: Y \to Y_{k-1}$ be the Postnikov approximation. Then $\pi(r_{k-1} \circ f) = 0$ and by 1.6 $r_k \circ f \approx w \circ \bar{\Delta}$ for some $w: X \wedge X \to Y_{k-1}$. By

1.4 $w$ can be lifted to $\bar{w}: X \wedge X \to Y$ and replacing $[f]$ by $[f] - [\bar{w} \circ \bar{\Delta}]$ if necessary $(\pi_m([f] - [\bar{w} \circ \bar{\Delta}]) = \pi_m([f]))$ we may assume that $r_{k-1} \circ f \approx *$; hence, $f$ can be lifted to $\bar{f}: X \to Y^k$, $f \approx i_k \circ \bar{f}$ ($i_k$: $Y^k \to Y$ the $k-1$ connective fibering). The proof is completed by observing the following diagram:




$$\pi_k(f) = \pi_k(i_k)(h_k^{(k)})^{-1} \circ H_k(\bar{f}) \circ h_k \in \operatorname{im} h_k^*.$$

Fig. 4

If $X$ satisfies (c) by a theorem of Serre the Hurewicz homomorphism induces a monomorphism $\tilde{h}_k$: $\pi_k(X)/\text{torsion} \to PH_k(X)/\text{torsion}$ (which is an isomorphism after tensoring with $Q$). If (d) is satisfied as well as (a), (b), and (c) one can talk about the degree of the Hurewicz-Serre homomorphism $\tilde{h}_k$.

As an immediate consequence of 1.7 one gets

COROLLARY 1.8. *Suppose* (a), (b), (c), *and* (d) *are satisfied. If* $f$: $X \to Y$ *satisfies* $\pi_m(f) = 0$ *for* $m < k$ *then* $\pi_k(f)x$ *is divisible by* $\deg \tilde{h}_k$ *where* $x$ *is either a generator of* $\pi_k(X)/$*torsion* $= Z$ *or* $x = 0$ *if* $\pi_k(X)/\text{torsion} = 0$.

We shall apply these propositions to $SU_{2n-1}$ (the Postnikov approximation of $SU$), $n \leqq \infty$:

PROPOSITION. 1.9. *Let* $n > 3$. *If* $f$:$SU_{2n-1} \to SU_{2n-1}$ *is a homotopy equiv-* $\deg \pi_{2n-1}$ *alence,* $\pi_k(f) = 1$, $k < 2n-1$ *then* $\pi_{2n-1}(f) = 1$.

PROOF. Let $[g] = [f] - 1$. Then $\pi_k(g) = 0$, $k < 2n-1$ and by 1.8 $(g) \equiv 0 \mod \deg \tilde{h}_{2n-1}$.

By 1.2 $\deg \tilde{h}_{2n-1} = (n-1)! > 2$ (if $n > 3$). As $f$ is a homotopy equivalence $|\deg \pi_{2n-1}(f)| = 1$ and, hence, $|\deg \pi_{2n-1}(g)| \leqq 2$. Consequently, $\deg \pi_{2n-1}(g$ $= 0$, $\deg \pi_{2n-1}(f) = 1$.

As every homotopy equivalence $SU_m \to SU_m$ induces a homotopy equivalence $SU_k \to SU_k$, $k \leqq m$, one gets:

Corollary 1.10. *If* $f: SU_m \to SU_m$ *is a homotopy equivalence,* $\deg \pi_3(f) = \deg \pi_5(f) = 1$ *then* $\pi_k(f) = 1$ *for all* $k$.

Proof. By induction, if $\pi_{2k-1}(f) = 1$, $3 \leqq k < n$, applying 1.9 to $f_n$: $SU_{2n-1} \to SU_{2n-1}$ induced by $f$, $\deg \pi_{2n-1}(f_n) = \deg \pi_{2n-1}(f) = 1$.

Lemma 1.11. *Let* $\chi(n): SU(n) \to SU(n)$ *be the map induced by complex conjugation. Let* $\chi: SU \to SU$ *be its limit. Then* $\deg \pi_{2n-1}(\chi) = (-1)^n$.

Proof. Three exists a commutative diagram

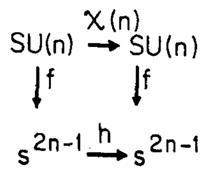



Fig. 5

where $f: SU(n) \to S^{2n-1}$ is the classical fibration and $\deg h = (-1)^n$. $\deg \pi_{2n-1}(f) \neq 0$ hence $\deg \pi_{2n-1}(h) . \deg \pi_{2n-1}(f) = \deg \pi_{2n-1}(f) . \deg \pi_{2n-1}(\chi(n))$ implies $\deg \pi_{2n-1}(\chi) = \deg \pi_{2n-1}(\chi(n)) = \deg \pi_{2n-1}(h) = (-1)^n$.

Note that $\chi$ (and hence its Postnikov approximation $\chi_n: SU_{2n-1} \to SU_{2n-1}$) is an $\infty$ loop map (and obviously a homotopy equivalence).

Theorem 1.12. $f: SU_{2n-1} \to SU_{2n-1}$ *is a homotopy equivalence if and only if* $[f] = [g] + \bar{\Delta}^* w$ *where* $g$ *is one of the four maps* $\pm 1$, $\pm \chi_n$, *and* $w \in [SU_{2n-1} \wedge SU_{2n-1}, SU_{2n-1}]$.

Proof. Clearly all maps $f$ satisfying $[f] = [g] + \bar{\Delta}^* w$ are homotopy equivalences.

If $f$ is a homotopy equivalence, then $g$ can be chosen among $\pm 1$, $\pm \chi_n$ so that $\pi_k(g \circ f) = 1$ for $k \leqq 5$. By 1.10 $\pi_k(g \circ f) = 1$ for all $k$ and by 1.6 $[g \circ f] - [1] \in \operatorname{im} \bar{\Delta}^*$. As $\pi(g \circ g \circ f) = \pi(f)$, $[f] - [g]^2[f] \in \operatorname{im} \bar{\Delta}^*$ (by 1.6) and as $\operatorname{im} \bar{\Delta}^*$ is a left ideal $[g][g \circ f] - [g][1] \in \operatorname{im} \bar{\Delta}^*$, hence

$$[f] - [g]^2[f] + [g]^2[f] - [g] = [f] - [g] \in \operatorname{im} \bar{\Delta}^*.$$

Remark 1.13. It could be easily seen that all homotopy equivalences of $SU_{2n-1}$ which are $H$-maps are homotopic to one of $\pm 1$, $\pm \chi_n$.

Corollary 1.14. *Every homotopy equivalence* $f_n: SU_{2n-1} \to SU_{2n-1}$ *can be covered by a homotopy equivalence* $f_m: SU_{2m-1} \to SU_{2m-1}$, $m > n$.

Proof. $\pm 1$ and $\pm \chi_n$ can be covered. If $w_n: SU_{2n-1} \wedge SU_{2n-1} \to SU_{2n-1}$ then $w_n(r_{2m-1,2n-1} \wedge r_{2m-1,2n-1})$: $SU_{2m-1} \wedge SU_{2m-1} \to SU_{2n-1}$

$(r_{2m-1,2n-1}: SU_{2m-1} \to SU_{2n-1})$ can be lifted (by 1.4) to $w_m: SU_{2m-1} \wedge SU_{2m-1} \to SU_{2m-1}$.

THEOREM 1.15. *If* $f: Sp_{4m-1} \to Sp_{4m-1}$ *is a homotopy equivalence then* $\deg \pi_{4k-1}(f) = \deg \pi_3(f)$ *for all* $k \geqq 1$.

PROOF. Consider $\phi_m: Sp_{4m-1} \to SU_{4m-1}$ induced by the Bott map $\phi$. Let $f: Sp_{4m-1} \to Sp_{4m-1}$ be a homotopy equivalence and suppose $\deg \pi_{4k-1}(f) = 1$, $1 \leqq k < n$. Then $\deg \pi_{4k-1}(\phi \circ f) = \deg \pi_{4k-1}(\phi)$, $k < n$. It follows from 1.8 that $\deg \pi_{4n-1}(\phi \circ f) \equiv \deg \pi_{4n-1}(\phi) \mod \deg \tilde{h}_{4n-1}$. As $n > 1$, by 1.2(d), $\deg \tilde{h}_{4n-1} \geqq 6$. But as $f$ is a homotopy equivalence $|\deg \pi_{4n-1}(f)| = 1$, hence

$$\deg \pi_{4n-1}(\phi \circ f) - \deg \pi_{4n-1}(\phi) \leqq 2 \deg \pi_{4n-1}(\phi) \leqq 4.$$

It follows that $\deg \pi_{4n-1}(\phi \circ f) = \deg \pi_{4n-1}(\phi)$ and as $\deg \pi_{4n-1}(\phi) \neq 0$ $\deg \pi_{4n-1}(f) = 1$. This proves that if $\deg \pi_3(f) = 1$ then $\deg \pi_{4k-1}(f) = 1$ for all $k$. If $\pi_3(f) = -1$ replace $[f]$ by $-[f]$.

## 2. Proof of Theorem A

PROPOSITION 2.1. *Let* $G, d, \tilde{\phi}_{dn-2}: G \to SU_{dn-2}$ *be either* $SU_{2n-2}, 2$, *identity or* $Sp_{4n-2}, 4, \phi_{4n-2}: Sp_{4n-2} \to SU_{4n-2}$ *induced by* $\phi$. *(In general, we write* $\tilde{\phi} = \phi_k: Sp_k \to SU_k$ *for the map induced by* $\phi: Sp \to SU$.*) If* $f, g: G \to G$ *are homotopy equivalences then there exists a homotopy equivalence* $h: SU_{dn-2} \to SU_{dn-2}$ *so that* $h \circ \tilde{\phi}_{dn-2} \circ f \approx \tilde{\phi}_{dn-2} \circ g$.

PROOF. If $d = 2$ and $h_1$ is the (homotopy) inverse of $f$ take $h = g \circ h_1$. If $d = 4$, in view of 1.15, by taking $[h_1] = \pm 1 \in [SU_{4n-2}, SU_{4n-2}]$ one may assume $\pi(\tilde{\phi} \circ f) = \pi(h_1 \circ \tilde{\phi} \circ g)$. By 1.6 $[\tilde{\phi} \circ f] - [h_1 \circ \tilde{\phi} \circ g] = \bar{\Delta}^*[w]$, $w: G \wedge G \to SU_{dn-2}$ As $\tilde{\phi} \wedge \tilde{\phi}$ (and consequently $\tilde{\phi} \circ g \wedge \tilde{\phi} \circ g$) induce an epimorphism of integral cohomology by 1.5 $w \approx \bar{w} \circ (\tilde{\phi} g \wedge \tilde{\phi} g)$, $\bar{w}: SU_{4n-2} \wedge SU_{4n-2} \to SU_{4n-2}$. Hence, $w \circ \bar{\Delta} \approx \bar{w} \circ \bar{\Delta} \circ \tilde{\phi} \circ g$. Put $[h] = [h_1] + \bar{\Delta}^*[\bar{w}]$ and then $[\tilde{\phi} \circ f] - [h \circ \tilde{\phi} \circ g] = [\tilde{\phi} \circ f] - [h_1 \circ \tilde{\phi} \circ g] - [\bar{w} \circ \bar{\Delta} \circ \tilde{\phi} \circ g] = 0$ and 2.1 follows.

As the fiber of $\tilde{h}_\lambda: M(n,d,\lambda) \to G(n,d)$ is $nd-2$ connected $r(\lambda): M(n,d,\lambda) \to G$, $(G, d, \tilde{\phi}$ as in 2.1) is a Postnikov approximation.

2.2. PROOF OF THEOREM A. Let $\alpha: M(n,d,\lambda) \to M(n,d,\lambda')$ be a homotopy equivalence. Let $f: M(n,d,\lambda) \to G(n,d)$ be the composition $\tilde{h}_{\lambda'} \circ \alpha$. Put $g = \tilde{h}_\lambda$. As noted above $r(\lambda): M(n,d,\lambda) \to G_{dn-2}$ and $r(\lambda'): M(n,d,\lambda') \to G_{dn-2}$ are Postnikov approximations and so is $r(\lambda') \circ \alpha$. Hence, one gets the following (homotopy) commutative diagram:

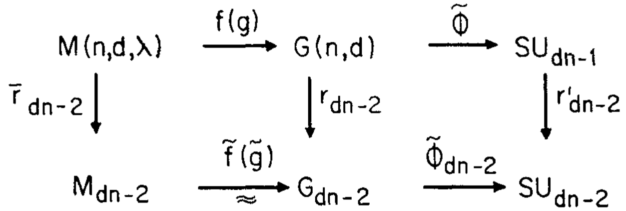



Fig. 6

$j$ and $\tilde{g}$ are homotopy equivalences. By 2.1 there exists $h\colon SU_{dn-2} \overset{\approx}{\to} SU_{dn-2}$ so that $h \circ \tilde{\phi}_{dn-2} \circ \tilde{f} \approx \tilde{\phi}_{dn-2} \circ \tilde{g}$. By 1.14 there exists $\bar{h}\colon SU_{dn-1} \to SU_{dn-1}$ with $r'_{dn-2} \circ h \approx h \circ r'_{dn-2}$. Hence, $r'_{dn-2} \circ \bar{h} \circ \tilde{\phi} \circ g \approx r'_{dn-2} \circ \tilde{\phi} \circ f$ and $[\bar{h} \circ \tilde{\phi} \circ g] - [\tilde{\phi} \circ f] \in \operatorname{im} j_*\colon [M(n,d,\lambda), K(dn-1,Z)] \to [M(n,d,\lambda), SU_{dn-1}]$ where $j\colon K(dn-1,Z) \to SU_{dn-1}$ is the fiber of $r'_{dn-2}$.

Let $\iota$, $w_{dn-1}$, $\tilde{w}_{dn-1}$, $v_{dn-1}$ be generators of the infinite cyclic groups $QH^{dn-1}(K(Z,dn-1),Z)$, $QH^{dn-1}(SU_{dn-1},Z)$, $QH^{dn-1}(G(n,d),Z)$ and $QH^{dn-1}(M(n,d,\lambda),Z)$ respectively then:

$$Qj^* w_{dn-1} = (dn/2-1)!\iota,$$

$$Q\tilde{\phi}^* w_{dn-1} = w_{dn-1}, \quad Q\bar{h}_\lambda^* w_{dn-1} = \lambda v_{dn-1}$$

$$Q\bar{h}^* w_{dn-1} = \pm w_{dn-1}, \quad Q(\bar{h}_{\lambda'} \circ \alpha)^* w_{dn-1} = \pm \lambda' v_{dn-1}.$$

As $[\bar{h} \circ \tilde{\phi} \circ g] - [\tilde{\phi} \circ f] \in \operatorname{im} j_*$,

$$Q(g^* \tilde{\phi}^* \bar{h}^*) w_{dn-1} \equiv Q(f^* \tilde{\phi}^*) w_{dn-1} \bmod (dn/2-1)\,!$$

and substituting the corresponding values $\lambda v_{dn-1} \geqq \pm \lambda' v_{dn-1} \bmod (dn/2-1)!$ and $\lambda \equiv \pm \lambda' \bmod (dn/2-1)!$,

### 3. Proof of Theorem B

Put $M(\lambda) = M(n,d,\lambda)$, $k = k(n,d)$, $G = G(n,d)$. If $\lambda \equiv \pm \lambda' \bmod k$ then $(\lambda,k) = (\lambda',k)$. As $\lambda/(\lambda,k)$ and $k/(\lambda,k)$ are relatively prime there exist $a,b \in Z$ so that

(1) $a(\lambda/(\lambda,k)) + b(k/(\lambda,k)) = 1$

(2) $(a, k/(\lambda,k)) = 1$

By (1) $(a\lambda'/(\lambda,k))\lambda \equiv \lambda' \bmod k$. Put $m = a\lambda'/(\lambda,k)$. Hence, there exists a commutative diagram

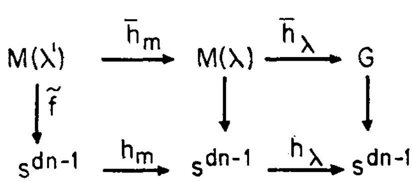



Fig. 7

Now, $h_k: S^{dn-1} \to S^{dn-1}$ can be lifted to $\chi: S^{dn-1} \to G$ and therefore $h_{k/(\lambda,k)}: S^{dn-1} \to S^{dn-1}$ can be lifted to $\chi': S^{dn-1} \to M(\lambda)$.

By (2) and since $\lambda'/(\lambda',k)$ and $k/(\lambda',k)$ are relatively prime $(m,k/(\lambda',k)) = 1$ and there exist integers $\alpha,\beta$ so that $\beta m - \alpha(k/(\lambda',k)) = \beta a\lambda'/(\lambda',k) - \alpha k/(\lambda',k) = 1$. As $k$ is even, $\lambda'$ odd$-k/(\lambda',k)$ is even and $m$ is odd, replacing $\alpha$, $\beta$ by $\alpha + m$, $\beta + k/(\lambda',k)$ if necessary one may assume that $\alpha$ is even.

Let $m: S^{dn-1} \times S^{dn-1} \to S^{dn-1}$ be a map of type (2,1). Define:

$$g_1: M(\lambda') \times S^{dn-1} \to M(\lambda), \ g_2: M(\lambda') \times S^{dn-1} \to S^{dn-1}$$

by

$$g_1 = \mu_{M(\lambda)}(\bar{h}_m \times \chi'), \ g_2 = m(h_{\alpha/2} \circ \tilde{f} \times h_\beta)$$

where $\mu_{M(\lambda)}$ is the $H$-structure of $M(\lambda)$ (which exists by [7]), $h_{\alpha/2}, h_\beta: S^{dn-1} \to S^{dn-1}$ maps of degrees $\alpha/2$ and $\beta$ respectively, $\tilde{f}: M(\lambda') \to S^{dn-1} \approx M(\lambda')/G$ the projection. Let $g: M(\lambda') \times S^{dn-1} \to M(\lambda) \times S^{dn-1}$ be induced by the $g_i$'s. Now, if $w_\lambda, w_{\lambda'}$, and $1_s$ are the generators of $QH^{dn-1}(M(\lambda),Z) \approx Z$, $QH^{dn-1}(M(\lambda'),Z)$, and $QH^{dn-1}(S^{dn-1},Z)$ respectively, then

$$QH^{dn-1}(g,Z)w_\lambda = mw_{\lambda'} + (k/(\lambda,k))1_s, \ QH^{dn-1}(g,Z)1_s = \alpha w_{\lambda'} + \beta 1_s$$

and $QH^{dn-1}(g,Z)$ is an isomorphism. As $QH^m(g,Z)$ is an isomorphism for $m < dn-1$, $QH^*(g, Z)$ is an isomorphism and as $H^*(M(\lambda) \times S^{dn-1},Z)$ and $H^*(M(\lambda') \times S^{dn-1},Z)$ are free (associative and commutative graded) algebras $H^*(g,Z)$ is an isomorphism and $g$ is a homotopy equivalence.

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